Math 261

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Optimization

1. **Define:** We say that f has a *critical point* at (a, b) if

either $f_x(a,b) = 0$ and $f_y(a,b) = 0$

or f is not differentiable at (a,b)

In other words, either $\nabla f(a,b) = \langle 0, 0 \rangle$ or $\nabla f(a,b)$ DNE.

2. The critical points are all the *potential* local maxima and minima.

3. **Second Derivative Test:** Suppose that f has a critical point at (a, b) and if f is differentiable on a disc around (a, b).

$$\text{Define } D = f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

There are four cases

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then f has a local min at (a,b) $f_{xx} > 0$

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then f has a local max at (a,b) $f_{xx} <$

(c) If $D < 0$, then f has a saddlepoint at (a,b) (no local max or mi

(d) If $D = 0$ then another test is required

4. To apply the second derivative test to multiple critical points, create the table

critical point	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx} f_{yy} - (f_{xy})^2$	analysis

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5. Find all critical points of the given function and use the second derivative test to identify each as a local maximum, a local minimum, or undetermined.

$$f(x, y) = x^2y - 4y$$

$f_x = 2xy$

$f_y = x^2 - 4$

f has critical pt if BOTH $= 0$ or one DNE

Both always exist

setting above $= 0$

$0 = 2xy$

$0 = x^2 - 4 = (x+2)(x-2)$

solving second gives $x = -2$ or $x = 2$

plugging into first gives

$0 = 4y \Rightarrow y = 0$

$0 = -4y \Rightarrow y = 0$

Critical points occur at $(-2, 0)$ and $(2, 0)$

Second derivative test

$f_{xx} = 2y$

$f_{yy} = 0$

$f_{xy} = 2x$

Critical pt	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx}f_{yy} - (f_{xy})^2$	analysis
$(-2, 0)$	0	0	-4	$0 \cdot 0 - (-4)^2 = -16$	$D < 0 \Rightarrow$ Saddle point (No local max or min.)
$(2, 0)$	0	0	4	$0 \cdot 0 - (4)^2 = -16$	$D < 0 \Rightarrow$ Saddle point

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6. Find all critical points of the given function and use the second derivative test to identify each as a local maximum, a local minimum, or undetermined.

$$f(x, y) = x^2 + y^3 + 6x - 3y$$

$f_x = 2x + 6$

$f_y = 3y^2 - 3$

critical pt when Both $= 0$ or one DNE.

Both always exist.

setting above $= 0$

$0 = 2x + 6$

$0 = 3y^2 - 3 = 3(y^2 - 1) = 3(y+1)(y-1)$

first $= 0 \Leftrightarrow 2x + 6 = 0 \Leftrightarrow x = -3$

second $= 0 \Leftrightarrow y = -1$ or $y = 1$

Critical points at $(-3, -1)$ and $(-3, 1)$

Second Derivative test

$f_{xx} = 2$

$f_{yy} = 6y$

$f_{xy} = 0$

Critical pt	f_{xx}	f_{yy}	f_{xy}	$D = f_{xx} f_{yy} - (f_{xy})^2$	analysis
$(-3, -1)$	2	-6	0	$2 \cdot (-6) - 0^2 = -12$	$D < 0 \Rightarrow$ saddle pt.
$(-3, 1)$	2	6	0	$2 \cdot 6 - 0^2 = 12$	$D > 0$ and $f_{xx} > 0 \Rightarrow$ local min $\cup$

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7. Find all critical points of the given function and use the second derivative test to identify each as a local maximum, a local minimum, or undetermined.

$$f(x,y) = -x^2 + xy - y^2 + 12y$$

$f_x = -2x + y$

$f_y = x - 2y + 12$

Critical pt when <u>BOTH</u> $= 0$ or <u>ONE</u> DNE.

Both always exist

Setting above $= 0$

$0 = -2x + y \iff y = 2x$

$0 = x - 2y + 12$

Solving first $y = 2x$

plugging first into second

$0 = x - 2(2x) + 12$

$0 = \underbrace{x - 4x}_{-3x} + 12$

$3x = 12$

$x = 6$

therefore $y = 2(x) = 12$

critical point at $(6,12)$

Second Derivative Test

$f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 1$

at $(6,12)$ $D = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - (1)^2 = 4 - 1 > 0$

and $f_{xx} < 0$

$\Rightarrow$ $\boxed{\text{f has a local max at } (6,12)}$